

ON KATZ'S BOUND FOR NUMBER OF ELEMENTS WITH GIVEN TRACE AND NORM

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ABSTRACT. In this note an improvement of the Katz's bound on the number of elements in a finite field with given trace and norm is given. The improvement is obtained by reducing the problem to estimating the number of rational points on certain toric Calabi-Yau hypersurface, and then to use detailed cohomological calculations by Rojas-Leon and the second author for such toric hypersurfaces.

1. INTRODUCTION

Let p be a prime and \mathbb{F}_q be the finite field of q elements of characteristic p . Given $a, b \in \mathbb{F}_q$, and positive integer $m \geq 2$, let

$$N_m(a, b) = \#\{\alpha \in \mathbb{F}_{q^m} \mid \text{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\alpha) = a, \text{Norm}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\alpha) = b\}.$$

Motivated by various applications, it is of interest to give a sharp estimate for the number $N_m(a, b)$. The case $b = 0$ is trivial.

Katz [2] proved the following bound:

Theorem 1.1. *Let $a, b \in \mathbb{F}_q^*$ and $n \geq 1$. Then*

$$|N_{n+1}(a, b) - \frac{q^{n+1} - 1}{q(q-1)}| \leq (n+1)q^{\frac{n-1}{2}}.$$

This bound was used by Moio [3] to improve some cases of the explicit bound in Wan [5] on the number of irreducible polynomials in an arithmetic progression of $\mathbb{F}_q[x]$. In the case $n+1 = 3$, the Katz bound also plays a significant role in Cohen and Huczynska [1] for their proof of the existence of a cubic primitive normal polynomial with given norm and trace.

If $a = 0$, Katz's bound can be improved in an elementary way using character sums [3]:

$$|N_{n+1}(0, b) - \frac{q^n - 1}{q - 1}| \leq (d-1)q^{\frac{n-1}{2}},$$

where $d = \gcd(n+1, q-1)$.

In this note, we give a uniform improvement of Katz's bound in the case $a \neq 0$.

Theorem 1.2. *Let $a, b \in \mathbb{F}_q^*$ and $n \geq 1$. Then*

$$|N_{n+1}(a, b) - \frac{q^n - 1}{q - 1}| \leq nq^{\frac{n-1}{2}}.$$

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In the case that $n + 1$ is a power of p , this improvement was first proved by Moio [3] using Deligne's estimate for hyper-Kloosterman sums. Moreover, in the case $n + 1 = 3$ also the bounds

$$3 \left\lfloor \frac{q + 1 - 2\sqrt{q}}{3} \right\rfloor \leq N_3(a, b) \leq 3 \left\lfloor \frac{q + 1 + 2\sqrt{q}}{3} \right\rfloor.$$

were obtained in [3] by using the Hasse's bound for elliptic curves together with a divisibility result. In corollary 2.4, we extend such divisibility bounds to $N_\ell(a, b)$, where $\ell \geq 3$ is any prime.

In the general case, our proof of Theorem 1.2 consists of two steps. The first step is to reduce it to estimating the number of \mathbb{F}_q -rational points on certain toric Calabi-Yau hypersurface over \mathbb{F}_q . The second step is to use the detailed cohomological calculations in Rojas-Leon and Wan [4] for such toric hypersurfaces. In the case $n + 1 = 3$, the above improved bounds should significantly reduce the amount of calculations in [1].

2. PROOF OF THEOREM 1.2

Let $u = b/a^{n+1} \in \mathbb{F}_q^*$. Let $N(u)$ denote the number of \mathbb{F}_q -rational points on the toric hypersurface

$$Y_u : X_1 + \cdots + X_n + \frac{u}{X_1 \cdots X_n} - 1 = 0.$$

Lemma 2.1.

$$N_{n+1}(a, b) = \frac{q^n - 1}{q - 1} + (-1)^n \left(N(u) - \frac{(q - 1)^n - (-1)^n}{q} \right).$$

Proof. Write the equation of Y_u in the form

$$\begin{aligned} X_1 + \cdots + X_{n+1} &= 1 \\ X_1 \cdots X_{n+1} &= u. \end{aligned}$$

Let ψ be the canonical additive character of \mathbb{F}_q . Now

$$q(q - 1)N(u) = \sum_{x_1, \dots, x_{n+1}} \sum_v \psi(v(x_1 + \cdots + x_{n+1} - 1)) \sum_\chi \chi(u^{-1}x_1 \cdots x_{n+1}),$$

where x_1, \dots, x_{n+1} run over \mathbb{F}_q^* , v runs over \mathbb{F}_q , and χ runs over the multiplicative character group of \mathbb{F}_q .

Let $G(\chi)$ denote the Gauss sum

$$G(\chi) = \sum_{x \in \mathbb{F}_q^*} \psi(x) \chi(x).$$

It follows that

$$\begin{aligned}
q(q-1)N(u) &= (q-1)^{n+1} + \sum_{v \neq 0} \psi(-v) \sum_{\chi} \bar{\chi}(u) \prod_{i=1}^{n+1} \sum_{x_i} \psi(vx_i) \chi(x_i) \\
&\stackrel{x_i \mapsto x_i/v}{=} (q-1)^{n+1} + \sum_{v \neq 0} \psi(-v) \sum_{\chi} \bar{\chi}(uv^{n+1}) G(\chi)^{n+1} \\
&= (q-1)^{n+1} + \sum_{\chi} G(\chi)^{n+1} \bar{\chi}(u) \sum_{v \neq 0} \psi(-v) \bar{\chi}^{n+1}(v) \\
(2.0.1) \quad &= (q-1)^{n+1} + \sum_{\chi} G(\chi)^{n+1} G(\bar{\chi}^{n+1}) \bar{\chi}((-1)^{n+1}u).
\end{aligned}$$

Next we express $N_{n+1}(a, b)$ in terms of Gauss sums. We use the abbreviated notations Tr and Norm in place of $\text{Tr}_{\mathbb{F}_{q^{n+1}}/\mathbb{F}_q}$ and $\text{Norm}_{\mathbb{F}_{q^{n+1}}/\mathbb{F}_q}$. Let $\psi_{n+1} = \psi \circ \text{Tr}$ be the canonical additive character of $\mathbb{F}_{q^{n+1}}$ and let $\alpha \in \mathbb{F}_{q^{n+1}}$ with $\text{Tr}(\alpha) = 1$. Now,

$$\begin{aligned}
q(q-1)N_{n+1}(a, b) &= \sum_{x \in \mathbb{F}_{q^{n+1}}^*} \sum_v \psi(v(\text{Tr}(x - \alpha a))) \sum_{\chi} \chi(b^{-1} \text{Norm}(x)) \\
&= \sum_v \psi(-av) \sum_{\chi} \bar{\chi}(b) \sum_x \psi_{n+1}(vx) \chi(\text{Norm}(x)) \\
&= q^{n+1} - 1 + \sum_{v \neq 0} \psi(-av) \sum_{\chi} \bar{\chi}(b) \sum_x \psi_{n+1}(vx) \chi(\text{Norm}(x)) \\
&\stackrel{x \mapsto x/v}{=} q^{n+1} - 1 + \sum_{v \neq 0} \psi(-av) \sum_{\chi} \bar{\chi}(bv^{n+1}) \sum_x \psi_{n+1}(x) \chi(\text{Norm}(x)),
\end{aligned}$$

since $\text{Norm}(v) = v^{n+1}$.

By the Davenport-Hasse identity the inner sum

$$\sum_x \psi_{n+1}(x) \chi(\text{Norm}(x)) = (-1)^n G(\chi)^{n+1},$$

and therefore

$$\begin{aligned}
q(q-1)N_{n+1}(a, b) &= q^{n+1} - 1 + (-1)^n \sum_{\chi} G(\chi)^{n+1} \bar{\chi}(b) \sum_{v \neq 0} \psi(-av) \bar{\chi}^{n+1}(v) \\
&= q^{n+1} - 1 + (-1)^n \sum_{\chi} G(\chi)^{n+1} G(\bar{\chi}^{n+1}) \bar{\chi}((-1)^{n+1}b/a^{n+1}).
\end{aligned}$$

Comparing this expression with (2.0.1), one finds that

$$N_{n+1}(a, b) = \frac{q^{n+1} - 1}{q(q-1)} + (-1)^n \left(N(u) - \frac{(q-1)^{n+1}}{q(q-1)} \right).$$

One checks that this is the same as the expression in Lemma 2.1. \square

This lemma reduces Theorem 1.2 to the following

Theorem 2.2. *Let $u \in \mathbb{F}_q^*$. Then*

$$\left| N(u) - \frac{(q-1)^n - (-1)^n}{q} \right| \leq nq^{\frac{n-1}{2}}.$$

Proof. Over the algebraic closure $\bar{\mathbb{F}}_q$, we can write $u = \lambda^{-(n+1)}$ for some non-zero element λ . Then Y_u is isomorphic to the toric hypersurface

$$X_\lambda : X_1 + \cdots + X_n + \frac{1}{X_1 \cdots X_n} - \lambda = 0$$

whose zeta function over a finite field was studied in detail in [4], see [6] for more elementary description of the results. For a prime $\ell \neq p$, the ℓ -adic cohomology

$$H_c^j(Y_u \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell) \cong H_c^j(X_\lambda \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell)$$

was calculated in Theorem 2.1 in [4]. In particular, we have

$$H_c^j(Y_u \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell) = 0, \quad j < n-1 \text{ or } j > 2n-1,$$

$$H_c^j(Y_u \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell^{\binom{j-n+2}{j-n+2}}(n-1-j), \quad n \leq j \leq 2n-2,$$

and there is an exact sequence of Galois modules

$$0 \rightarrow \mathbb{Q}_\ell^n \rightarrow H_c^{n-1}(Y_u \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell) \rightarrow M_u \rightarrow 0,$$

where M_u is of rank at most n and mixed of weight at most $n-1$. It follows that

$$|\mathrm{Tr}(\mathrm{Frob}_u | M_u)| \leq nq^{\frac{n-1}{2}}.$$

By the ℓ -adic trace formula,

$$N(u) = \sum_{j=n}^{2n-2} (-1)^j \binom{n}{j-n+2} q^{j-(n-1)} + (-1)^{n-1}n + (-1)^{n-1}\mathrm{Tr}(\mathrm{Frob}_u | M_u).$$

Replacing j by $j+n-2$, one finds

$$N(u) = \sum_{j=2}^n (-1)^{j-n} \binom{n}{j} q^{j-1} + (-1)^{n-1}n + (-1)^{n-1}\mathrm{Tr}(\mathrm{Frob}_u | M_u).$$

The theorem follows. \square

Remark. If $u \neq (n+1)^{-(n+1)}$, i.e., $\lambda \notin \{(n+1)\zeta | \zeta^{n+1} = 1\}$, then M_u is pure of weight $n-1$ and of rank n . If $u = (n+1)^{-(n+1)}$ (necessarily $p \nmid n+1$), then the rank of M_u drops by 1 and thus

$$|\mathrm{Tr}(\mathrm{Frob}_u | M_u)| \leq (n-1)q^{\frac{n-1}{2}}.$$

If $u = (n+1)^{-(n+1)}$ and n is even, then one of the Frobenius eigenvalues has weight $n-2$ (instead of $n-1$), and thus

$$|\mathrm{Tr}(\mathrm{Frob}_u | M_u)| \leq (n-2)q^{\frac{n-1}{2}} + q^{\frac{n-2}{2}}.$$

All these follow from Proposition 2.6 in [4].

Corollary 2.3. *Let $u = (n+1)^{-(n+1)}$. Then*

$$|N(u) - \frac{(q-1)^n - (-1)^n}{q}| \leq (n-1)q^{\frac{n-1}{2}}.$$

If n is also even, then

$$|N(u) - \frac{(q-1)^n - (-1)^n}{q}| \leq (n-2)q^{\frac{n-1}{2}} + q^{\frac{n-2}{2}}.$$

Corollary 2.4. *Let $\ell \geq 3$ be a prime number. Let $a, b \in \mathbb{F}_q^*$. Then, we have*

$$\ell \left\lfloor \frac{\frac{q^{\ell-1}-1}{q-1} - (\ell-1)q^{(\ell-2)/2}}{\ell} \right\rfloor \leq N_\ell(a, b) \leq \ell \left\lfloor \frac{\frac{q^{\ell-1}-1}{q-1} + (\ell-1)q^{(\ell-2)/2}}{\ell} \right\rfloor.$$

Proof. Let R be the number of $c \in \mathbb{F}_q$ such that $\ell c = a$ and $c^\ell = b$. It is clear that R is either 0 or 1. Since ℓ is a prime, $N_\ell(a, b) - R$ is divisible by ℓ . If $R = 0$, the corollary is the consequence of Theorem 1.2 and the divisibility of $N_\ell(a, b)$ by ℓ .

Assume now that $R = 1$. Since $a \neq 0$, ℓ cannot be p . In this case, we have $a = \ell c$, $b = c^\ell$ and thus $u = b/a^\ell = \ell^{-\ell} \in \mathbb{F}_q^*$. We can apply the stronger estimate in the previous corollary to deduce the desired inequalities for $N_\ell(a, b)$.

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